1. $\Delta u = \cos\varphi$, $r > 3$,
   $u(3,\varphi) = 3\sin 2\varphi$.

2. $\Delta u = 0$, $0 < x < \pi, 0 < y < \frac{\pi}{2}$, если $u(0,y) = 0, u(\pi,y) = 0, u(x,0) = 0, u(x,\frac{\pi}{2}) = \sin 3x$

3. $\Delta u = 0, -\infty < x < \infty, -2\pi < y < 0; u(x,-2\pi) = 0,\ u(x,0) = 1,$ при $x < 1; u(x,0) = 0,$ при $x \geq 1$

4. $$\begin{cases} u_{tt} = u_{xx} - \sin 3x, t > 0, 0 < x < \frac{\pi}{2}; \\ u(0,t) = t, \quad u_x(\frac{\pi}{2},t) = 0, \\ u(x,0) = 2\sin 3x - \sin x, \\ u_t(x,0) = \sin x + 1. \end{cases}$$

5. $u_{tt} = 9u_{xx}, 0 < x < \infty, t > 0$

$u_x(0,t) = 0$, $u(x,0) = \begin{cases} \sin x, x \in [0;\pi] \\ 0, x \notin [0;\pi] \end{cases}$, $u_t(x,0) = 0$. Найти $u(\pi/4, t)$ и нарисовать график.

6. $u_{tt} = 4u_{xx}, t > 0, x > 0;$

$u_x(0,t) = 0 \quad u(x,0) = \sin x \quad u_t(x,0) = 2\cos x.$

Описать процесс колебаний. Построить профиль струны в момент времени $t = \frac{\pi}{4}$.

$$\begin{cases} \Delta u = \cos\varphi, & r > 3 \\ u(3,\varphi) = 3\sin 2\varphi \end{cases}$$ вне круга

$u_{част} = A r^2 \cos\varphi$

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \cos\varphi$$

$(2r^2)' = 4r$

$\cancel{\frac{1}{r} A \cos\varphi \left(\frac{\partial}{\partial r}\right)}$ $\frac{1}{r} A\cos\varphi\,(r\cdot 2r)' - \frac{1}{r^2} A r^2 \cos\varphi = \cos\varphi$

$4A - A = 1 \Rightarrow A = \frac{1}{3}$

$u_{част} = \frac{r^2}{3}\cos\varphi$ $\qquad v = u - u_{част}$

$$\begin{cases} \Delta v = 0 \\ v(3,\varphi) = 3\sin 2\varphi + 3\cos\varphi \end{cases}$$

$$\underset{\substack{общ \\ вне\ круга}}{v(r,\varphi)} = A + \sum_{n=1}^{\infty} r^{-n}(a_n \cos n\varphi + b_n \sin n\varphi) \Rightarrow$$

$\left(\frac{1}{3}\right) a_n = 3 \Rightarrow a_n = 9$

$\left(\frac{1}{3}\right)^2 b_n = 3 \Rightarrow b_n = 27$

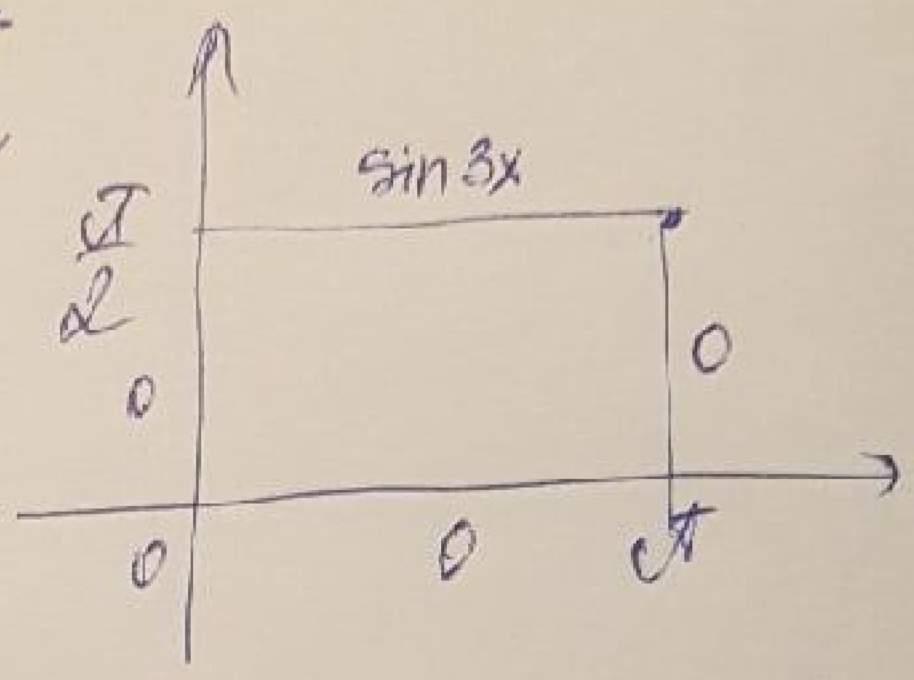
$\Rightarrow v(r,\varphi) = \frac{9\cos\varphi}{r} + \frac{27\sin 2\varphi}{r^2}$

$\Rightarrow u(r,\varphi) = \frac{9\cos\varphi}{r} + \frac{27\sin 2\varphi}{r^2} + \frac{r^2}{3}\cos\varphi$

не уверен т.к. $\underset{r\to\infty}{\longrightarrow} \infty$

№2

$$\begin{cases} \Delta u = 0, \; 0 < x < \pi, \; 0 < y < \frac{\pi}{2} \\ u(0,y) = 0 \\ u(\pi,y) = 0 \\ u(x,0) = 0 \\ u(x, \pi/2) = \sin 3x \end{cases}$$



$u = X(x)\,Y(y)$

$X'' + \lambda X = 0$

$X(0) = 0 \quad \Rightarrow \{I - I\} \Rightarrow \quad \lambda_n = n^2$

$X(\pi) = 0 \qquad\qquad\qquad\qquad X_n = \sin nx, \; n \in \mathbb{N}$

$Y'' - \lambda Y = 0$

$Y_n = A_n e^{ny} + B_n e^{-ny}$

$$\begin{cases} u(x,y) = \sum_{n=1}^{\infty} (A_n e^{ny} + B_n e^{-ny}) \sin nx \\ u(x,0) = 0 = \sum_{n=1}^{\infty} (A_n + B_n) \sin nx \\ u(x, \frac{\pi}{2}) = \sin 3x = \sum_{n=1}^{\infty} (A_n e^{n\frac{\pi}{2}} + B_n e^{-n\frac{\pi}{2}}) \sin nx \end{cases}$$

$n \neq 3 \Rightarrow \begin{cases} A_n + B_n = 0 \\ A_n e^{n\frac{\pi}{2}} + B_n e^{-n\frac{\pi}{2}} = 0 \end{cases} \Rightarrow \begin{matrix} A_n = 0 \\ B_n = 0 \end{matrix}$

$n = 3 \Rightarrow \begin{cases} A_3 + B_3 = 0 \\ A_3 e^{\frac{3\pi}{2}} + B_3 e^{-\frac{3\pi}{2}} = 1 \end{cases} \quad \begin{matrix} A_3 = -B_3 \\ B_3 (e^{-\frac{3\pi}{2}} - e^{\frac{3\pi}{2}}) = 1 \end{matrix}$

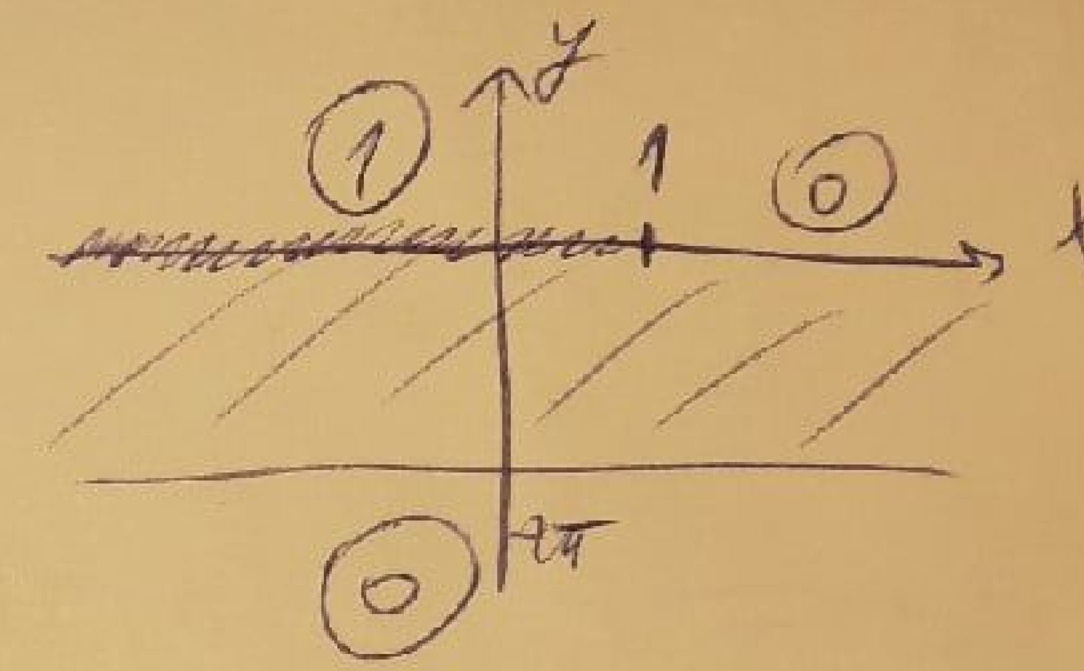
$A_3 = \frac{1}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}}$

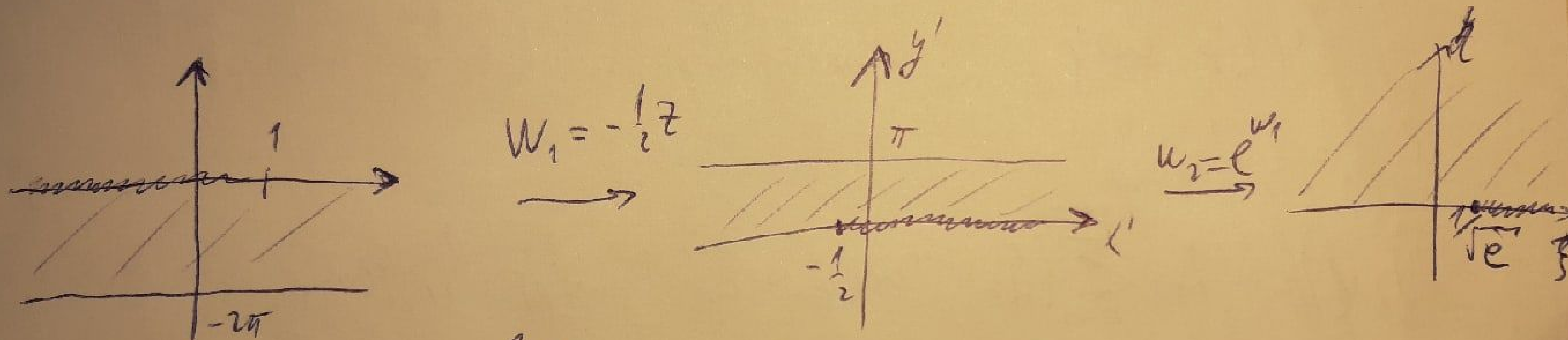
$B_3 = \frac{1}{e^{-\frac{3\pi}{2}} - e^{\frac{3\pi}{2}}}$

$$\Rightarrow \boxed{u(x,y) = \left(\frac{e^{3y}}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}} + \frac{e^{-3y}}{e^{-\frac{3\pi}{2}} - e^{\frac{3\pi}{2}}} \right) \sin 3x}$$

В. IV №3.

$$\begin{cases} \Delta u = 0, & -\infty < x < +\infty,\ -2\pi < y < 0 \\ u(x,-2\pi) = 0 \\ u(x,0) = \begin{cases} 1, & x < 1 \\ 0, & x \geq 1 \end{cases} \end{cases}$$





Итого $w = e^{-\frac{1}{2}z}$

$$z = x+iy \Rightarrow w = e^{-\frac{1}{2}(x+iy)} = e^{-\frac{1}{2}x}\cdot e^{-\frac{1}{2}iy} = e^{-\frac{1}{2}x}\left(\cos\left(-\frac{y}{2}\right) + i\sin\left(-\frac{y}{2}\right)\right) =$$

$$= e^{-\frac{1}{2}x}\left(\cos\frac{y}{2} - i\sin\frac{y}{2}\right) \Rightarrow \xi_0 = e^{-\frac{1}{2}x_0}\cos\frac{y_0}{2},\ \eta_0 = e^{-\frac{1}{2}x_0}\left(-\sin\frac{y_0}{2}\right)$$

$$u \xrightarrow{w} V: \begin{cases} \Delta V = 0 \\ V(\xi,0) = \begin{cases} 0, & \xi \leq \frac{1}{\sqrt{e}} \\ 1, & \xi > \frac{1}{\sqrt{e}} \end{cases} = f(\xi) \end{cases}$$

$$V(w_0) = \frac{\eta_0}{\pi}\int_{-\infty}^{+\infty} \frac{f(\xi)\,d\xi}{(\xi-\xi_0)^2+\eta_0^2} = \frac{\eta_0}{\pi}\int_{1/\sqrt{e}}^{+\infty}\frac{d\xi}{(\xi-\xi_0)^2+\eta_0^2} = \frac{1}{\pi}\operatorname{arctg}\frac{\xi-\xi_0}{\eta_0}\Big|_{\frac{1}{\sqrt{e}}}^{+\infty} =$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \operatorname{arctg}\frac{\frac{1}{\sqrt{e}}-\xi_0}{\eta_0}\right]$$

$$u(z_0) = \frac{1}{\pi}\left[\frac{\pi}{2} - \operatorname{arctg}\frac{\frac{1}{\sqrt{e}} - e^{-\frac{1}{2}x_0}\cos\frac{y_0}{2}}{-e^{-\frac{1}{2}x_0}\sin\frac{y_0}{2}}\right]$$

$$\begin{cases} u_{tt} = u_{xx} - \sin 3x & t>0,\ 0<x<\frac{\pi}{2} \\ u(0,t) = t \\ u_x(\frac{\pi}{2}, t) = 0 \\ u(x,0) = 2\sin 3x - \sin x \\ u_t(x,0) = \sin x + 1 \end{cases}$$

обнул. гр. условия. $w(x,t) = t$, $u(x,t) = v(x,t) + w(x,t)$

$$\begin{cases} v_{tt} = v_{xx} - \sin 3x \\ v(0,t) = 0 \\ v_x(\frac{\pi}{2}, t) = 0 \\ v(x,0) = 2\sin 3x - \sin x \\ v_t(x,0) = \sin x \end{cases}$$

$\left.\begin{matrix} v(0,t)=0 \\ v_x(\frac{\pi}{2},t)=0 \end{matrix}\right\}$ гр. усл. I – II $\Rightarrow$ $\lambda_n = (2n-1)^2$, $n \in \mathbb{N}$

$X_n = \sin(2n-1)x$

Можно искать $v(x,t)$ в виде $A(t)\sin x + B(t)\sin 3x$

$$A''\sin x + B''\sin 3x = -A\sin x - 9B\sin 3x + \sin 3x$$

$A(0) = -1$ $\quad B(0) = 2$

$A'(0) = 1$ $\quad B'(0) = 0$

$$\begin{cases} A'' = -A \\ A(0) = -1 \\ A'(0) = 1 \end{cases} \Rightarrow A_{общ}(t) = C_1\cos t + C_2\sin t$$

$A(0) = C_1 = -1$

$A'(0) = C_2 = 1$

$A(t) = (\sin t - \cos t)$

$$\begin{cases} B'' = -9B + 1 \\ B(0) = 2 \\ B'(0) = 0 \end{cases}$$

$B_{однор}(t) = C_1\cos 3t + C_2\sin 3t$

$B_{част}(t) = a$:

$0 = -9a + 1 \Rightarrow a = \frac{1}{9}$

$B_{общ} = C_1\cos 3t + C_2\sin 3t + \frac{1}{9}$

$B(0) = C_1 + \frac{1}{9} = 2 \Rightarrow C_1 = \frac{17}{9}$

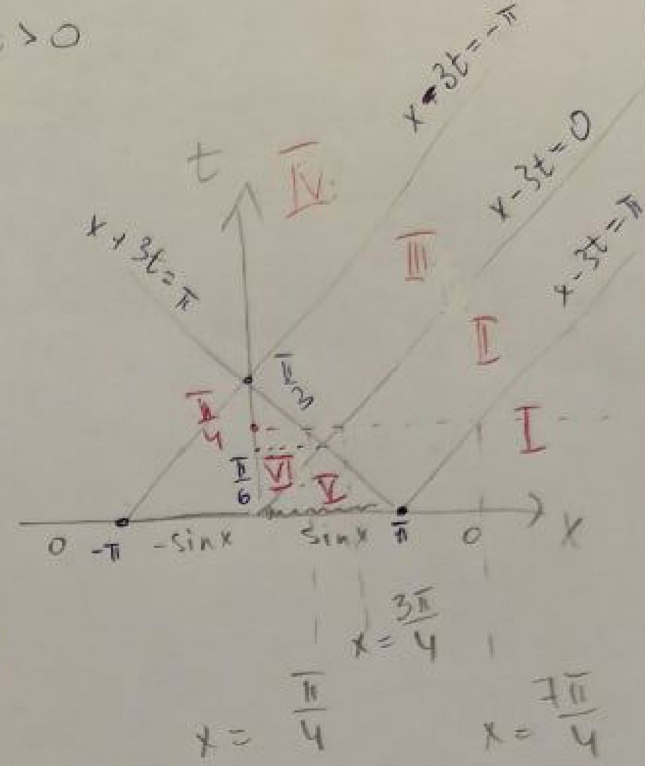
$B'(0) = 3C_2 = 0 \Rightarrow C_2 = 0$

$v(x,t) + t = u(x,t)$

$$u(x,t) = (\sin t - \cos t)\cdot\sin x + \left(\frac{17}{9}\cos 3t + \frac{1}{9}\right)\sin 3x$$

5 $u_{tt} = 9u_{xx} \quad 0<x<\infty,\ t>0$

$$\begin{cases} u_x(0,t) = 0 \\ u(x,0) = \begin{cases} \sin x, & x\in[0,\pi] \\ 0, & x\notin[0,\pi] \end{cases} \\ u_t(x,0) = 0 \end{cases}$$



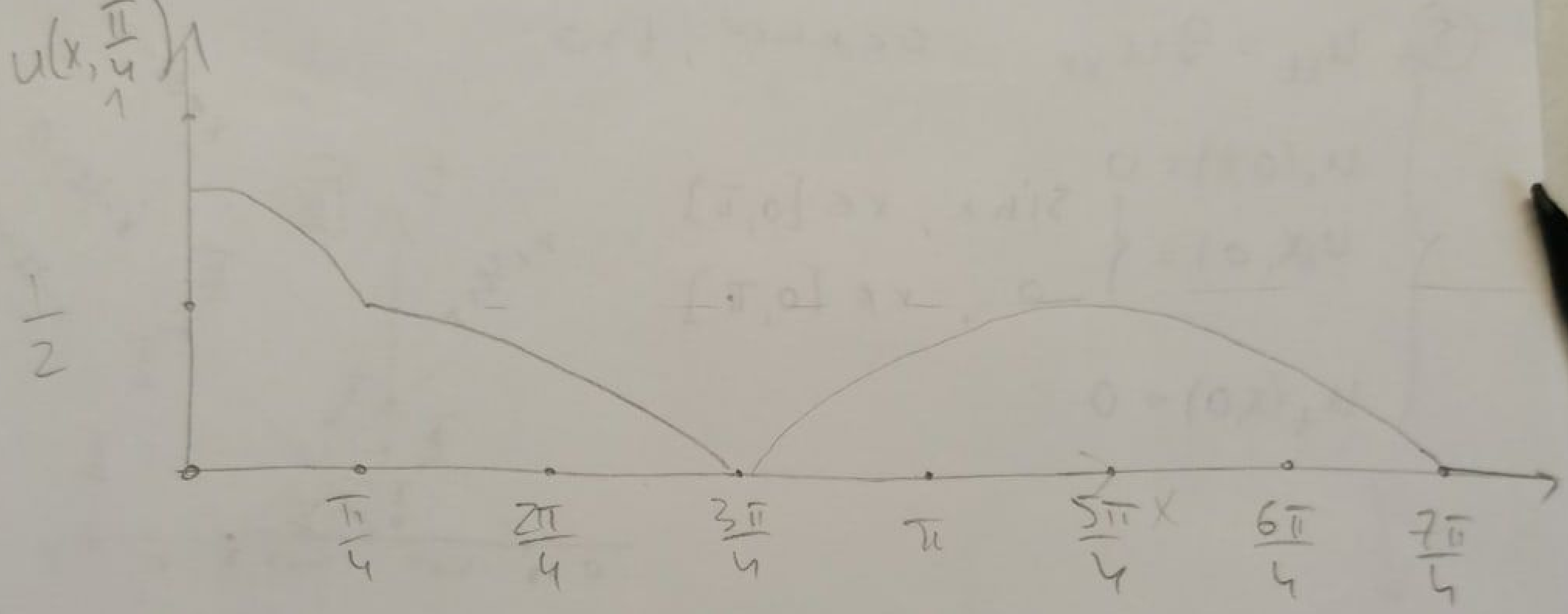
$\downarrow$

$u_{tt} = 9u_{xx}, \quad x\in\mathbb{R},\ t>0$

$$\begin{cases} u(x,0) = \begin{cases} \sin x, & x\in[0,\pi] \\ -\sin x, & x\in[-\pi,0) \\ 0, & x\notin[-\pi,\pi] \end{cases} = \varphi(x) \\ u_t(x,0) = 0 \end{cases}$$

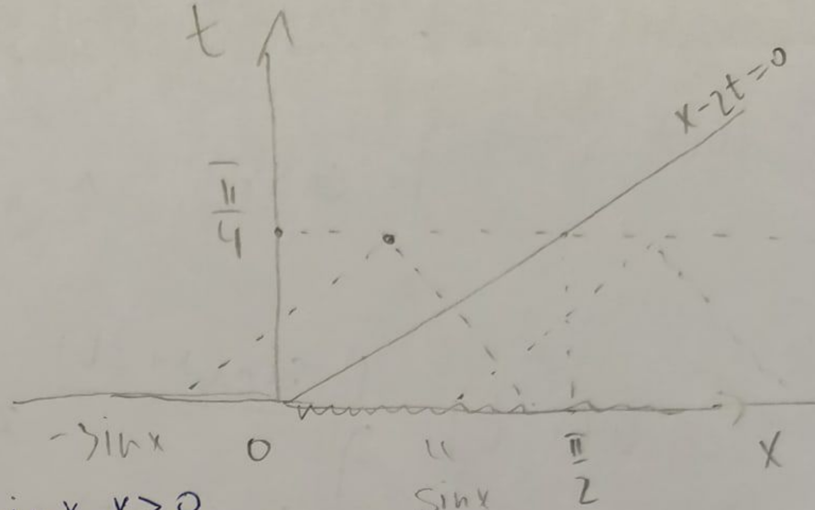
$$u(x,t) = \frac{\varphi(x-3t)+\varphi(x+3t)}{2}$$

$$u\left(x,\frac{\pi}{4}\right) = \begin{cases} \dfrac{-\sin\left(x-\frac{3\pi}{4}\right)+\sin\left(x+\frac{3\pi}{4}\right)}{2}, & 0\le x<\frac{\pi}{4} & \text{VI} \\ \dfrac{-\sin\left(x-\frac{3\pi}{4}\right)+0}{2}, & \frac{\pi}{4}<x\le\frac{3\pi}{4} & \text{III} \\ \dfrac{\sin\left(x-\frac{3\pi}{4}\right)+0}{2}, & \frac{3\pi}{4}<x\le\frac{7\pi}{4} & \text{II} \\ 0, & x>\frac{7\pi}{4} & \text{I} \end{cases}$$



$u(x, \frac{\pi}{4})$
1
$\frac{1}{2}$
$\frac{\pi}{4}$
$\frac{2\pi}{4}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{6\pi}{4}$
$\frac{7\pi}{4}$
x

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < \infty,\ t > 0 \\ u_x(0,t) = 0 \\ u(x,0) = \sin x = \varphi(x) \\ u_t(x,0) = 2\cos x = \psi(x) \end{cases}$$



$\downarrow$

$$\begin{cases} u_{tt} = 4u_{xx} \\ u(x,0) = \Phi(x) = \begin{cases} \sin x, & x > 0 \\ -\sin x, & x < 0 \end{cases} \\ u_t(x,0) = \Psi(x) = 2\cos x \end{cases}$$

$$u(x,t) = \frac{\Phi(x-2t) + \Phi(x+2t)}{2} + \frac{1}{2a}\int_{x-2t}^{x+2t} \Psi(\alpha)\,d\alpha .$$

$$u\left(x,\frac{\pi}{4}\right) = \frac{\Phi\left(x-\frac{\pi}{2}\right) + \Phi\left(x+\frac{\pi}{2}\right)}{2} + \frac{1}{2a}\int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} 2\cos\alpha\,d\alpha =$$

$$= \frac{1}{a}\sin\alpha\Big|_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} = \frac{2\cos x}{a}, \quad a = 2$$

$x \in \left[0, \frac{\pi}{2}\right]$:

$$u\left(x,\frac{\pi}{4}\right) = \underbrace{\frac{-\sin\left(x-\frac{\pi}{2}\right) + \sin\left(x+\frac{\pi}{2}\right)}{2}}_{=\cos 2x} + \frac{1}{2a}\int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} 2\cos\alpha\,d\alpha = 2\cos x$$

$x \geq \frac{\pi}{2}$:

$$u\left(x, \frac{\pi}{4}\right) = \underbrace{\frac{\sin\left(x - \frac{\pi}{2}\right) + \sin\left(x + \frac{\pi}{2}\right)}{2}}_{=0} + \overbrace{\frac{1}{2a} \int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} 2\cos\alpha \, d\alpha}^{=\cos x} = \cos x$$

$$u\left(x, \frac{\pi}{4}\right) = \begin{cases} 2\cos x, & x \in \left[0, \frac{\pi}{2}\right) \\ \cos x, & x \in \left[\frac{\pi}{2}, +\infty\right) \end{cases}$$

$$u(x,t) = \begin{cases} \dfrac{\sin(x+2t) - \sin(x-2t)}{2} + \dfrac{1}{4} \displaystyle\int_{x-2t}^{x+2t} 2\cos\alpha \, d\alpha, & x \in \left[0, \frac{\pi}{2}\right) \\ \dfrac{\sin(x+2t) + \sin(x-2t)}{2} + \dfrac{1}{4} \displaystyle\int_{x-2t}^{x+2t} 2\cos\alpha \, d\alpha, & x \geq \frac{\pi}{2} \end{cases}$$